Regd. Office: 2nd Floor, Grand Plaza, Fraser Road, Dak Bunglow, Patna - 800001

JEE Main 2023 (Memory based)

25 January 2023 - Shift 1

Answer & Solutions

MATHEMATICS

1. $\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}$, $x\in[-1,\ 1]$ sum of all solutions is $\alpha-\frac{4}{\sqrt{3}}$, then α is:

A. 1
B. 2
C. −2
D. $\sqrt{3}$

Answer (B)

Solution:

$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{3}$

for $-1<x<0$, $\tan^{-1}\left(\frac{2x}{1-x^2}\right)=2\tan^{-1}x$ and $\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\pi+2\tan^{-1}x$

$2\tan^{-1}x+\pi+2\tan^{-1}x=\frac{\pi}{3}$

$4\tan^{-1}x=-\frac{2\pi}{3}$

$x=-\frac{1}{\sqrt{3}}$

for $0<x<1$, $\tan^{-1}\left(\frac{2x}{1-x^2}\right)=2\tan^{-1}x$ and $\cot^{-1}\left(\frac{1-x^2}{2x}\right)=2\tan^{-1}x$

$4\tan^{-1}x=\frac{\pi}{3}$

$x=\tan\frac{\pi}{12}=2-\sqrt{3}$

sum $=2-\sqrt{3}-\frac{1}{\sqrt{3}}=2-\frac{4}{\sqrt{3}}$

$\therefore\alpha=2$

2. Mean of a data set is 10 and variance is 4. If one entry of data set changes from 8 to 12, then new mean becomes 10.2. Then now variance is:

A. 3.92
B. 3.96
C. 4.04
D. 4.08

Answer (B)

Solution:

Let number of observations be n

$10n-8+12=(10.2)n$

$10n+4=(10.2)n$

$\Rightarrow n=20$

For earlier set of observations

$\frac{\sum x_i^2}{20}-(10)^2=4$

$\Rightarrow\sum x_i^2=(104)(20)=2080$

After change

$(\sum x_i^2)_{\text{new}}=2080-8^2+12^2$

$=2160$

New variance $=\frac{2160}{20}-(10.2)^2$

$=108-(10.2)^2$

$=3.96$

3. If $y = (1 + x)(x^2 + 1)(x^4 + 1)(x^8 + 1)(x^{16} + 1)$, then find the value of $y'' - y'$ at $x = -1$:

A. 496
B. 946
C. -496
D. -946

Answer (C)

Solution:

$y = (1 + x)(x^2 + 1)(x^4 + 1)(x^8 + 1)(x^{16} + 1)$
Multiply and divide by $(x - 1)$ we get,
$y = \frac{(1+x)(x^2+1)(x^4+1)(x^8+1)(x^{16}+1)(x-1)}{(x-1)}$
$\Rightarrow y = \frac{(x^2-1)(x^2+1)(x^4+1)(x^8+1)(x^{16}+1)}{(x-1)}$
$\Rightarrow y = \frac{(x^4-1)(x^4+1)(x^8+1)(x^{16}+1)}{(x-1)}$
$\Rightarrow y = \frac{(x^8-1)(x^8+1)(x^{16}+1)}{(x-1)}$
$\Rightarrow y = \frac{(x^{16}-1)(x^{16}+1)}{(x-1)}$
$\Rightarrow y = \frac{(x^{32}-1)}{(x-1)}$
At $x = -1$ we get $y = 0$
$y(x - 1) = x^{32} - 1$
Differentiate on both sides,
$y'(x - 1) + y = 32x^{31} \quad \cdots (1)$
At $x = -1$
$y'(-1) = \frac{-32}{-2} = 16$
Differentiate equation (1) on both sides we get,
$y''(x - 1) + y' + y' = 32 \times 31x^{30}$
At $x = -1$
$y''(-1) = \frac{32\times31-16-16}{-2} = -480$
$\therefore y''(-1) - y'(-1) = -480 - 16 = -49$

4. The logical statement $(p \wedge \sim q) \rightarrow (p \rightarrow \sim q)$ is a:

A. Tautology
B. Fallacy
C. Equivalent to $p \vee \sim q$
D. Equivalent to $p \wedge \sim q$

Answer (A)

Solution:

$(p \wedge \sim q) \rightarrow (p \rightarrow \sim q)$
$= (p \wedge \sim q) \rightarrow (\sim p \vee \sim q)$
$= \sim(p \wedge \sim q) \vee (\sim p \vee \sim q)$
$= (\sim p \vee q) \vee (\sim p \vee \sim q)$
$= \sim p \wedge T = T$ (Tautology)

5. If a_r is the coefficient of x^{10-r} in expansion of $(1 + x)^{10}$ then $\sum_{r=1}^{10} r^3 \left(\frac{a_r}{a_{r-1}}\right)^2$ is:

A. 390
B. 1210
C. 485
D. 220

Answer (B)

Solution:

Coefficient of x^{10-r} in $(1+x)^{10}$ is ${}^{10}C_{10-r}$

$\therefore a_r = {}^{10}C_{10-r}$

$\sum_{r=1}^{10} r^3\left(\frac{a_r}{a_{r-1}}\right)^2 = \sum_{r=1}^{10} r^3 \cdot \left(\frac{10!}{r!(10-r)!}\cdot\frac{(11-r)!(r-1)!}{10!}\right)^2$

$= \sum_{r=1}^{10} r^3 \cdot \left(\frac{11-r}{r}\right)^2 = \sum_{r=1}^{10} r(11-r)^2$

$\sum_{r=1}^{10} r(11-r)^2 = 1\times 10^2 + 2\times 9^2 + \cdots + 9\times 2^2 + 10\times 1^2$

Which is same as $\sum_{r=1}^{10} r^2(11-r)$

$\sum_{r=1}^{10} r^2(11-r) = 1^2\times 10 + 2^2\times 9 + \cdots + 9^2\times 2 + 10^2\times 1$

$\Rightarrow \sum_{r=1}^{10} r(11-r)^2 = \sum_{r=1}^{10} r^2(11-r)$

$\Rightarrow \sum_{r=1}^{10} r^2(11-r) = 11\sum_{r=1}^{10} r^2 - \sum_{r=1}^{10} r^3$

$\Rightarrow \sum_{r=1}^{10} r^3\left(\frac{a_r}{a_{r-1}}\right)^2 = 11\left(\frac{10\times 11\times 21}{6}\right) - \left(\frac{10\times 11}{2}\right)^2$

$\Rightarrow \sum_{r=1}^{10} r^3\left(\frac{a_r}{a_{r-1}}\right)^2 = 11^2\times 35 - 11^2\times 25$

$\Rightarrow \sum_{r=1}^{10} r^3\left(\frac{a_r}{a_{r-1}}\right)^2 = 11^2\times 10 = 1210$

6. $\lim\limits_{n\to\infty}\frac{1+2-3+4+5-6+\cdots(3n-2)+(3n-1)-3n}{\sqrt{2n^4+3n+1}-\sqrt{n^4+n+3}}$

A. $\frac{3}{2}(\sqrt{2}+1)$
B. $\frac{2}{3}(\sqrt{2}+1)$
C. $\frac{2}{3\sqrt{2}}$
D. $2\sqrt{2}$

Answer (A)

Solution:

$\lim\limits_{n\to\infty}\frac{1+2-3+4+5-6+\cdots(3n-2)+(3n-1)-3n}{\sqrt{2n^4+3n+1}-\sqrt{n^4+n+3}}$

$= \lim\limits_{n\to\infty}\frac{\sum_{r=1}^{n}\left((3r-2)+(3r-1)-3r\right)}{\sqrt{2n^4+3n+1}-\sqrt{n^4+n+3}}$

$= \lim\limits_{n\to\infty}\frac{\sum_{r=1}^{n}3(r-1)}{\sqrt{2n^4+3n+1}-\sqrt{n^4+n+3}}$

$= \lim\limits_{n\to\infty}\frac{3\frac{n(n-1)}{2}}{\sqrt{2n^4+3n+1}-\sqrt{n^4+n+3}}$

$= \lim\limits_{n\to\infty}\frac{3\frac{n(n-1)}{2}}{n^2\left(\sqrt{2+\frac{3}{n^3}+\frac{1}{n^4}}-\sqrt{1+\frac{1}{n^3}+\frac{3}{n^4}}\right)}$

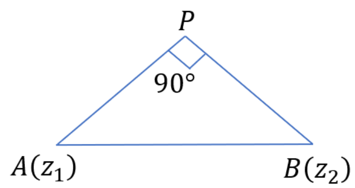
$= \frac{3}{2}\left(\frac{1}{\sqrt{2}-1}\right) = \frac{3}{2}(\sqrt{2}+1)$

7. If $|z-z_1|^2 + |z-z_2|^2 = |z_1-z_2|^2$ when $z_1 = 2+3i$ and $z_2 = 3+4i$, then locus of z is:

A. Straight line with slope $-\frac{1}{2}$
B. Circle with radius $\frac{1}{\sqrt{2}}$
C. Hyperbola with eccentricity $\sqrt{2}$
D. Hyperbola with eccentricity $\frac{5}{2}$

Answer (B)

Solution:



So, locus of P is circle whose diameter is AB
$AB = \sqrt{2}$
$\therefore$ radius of circle $= \frac{1}{\sqrt{2}}$

8. $f(x) = \int \frac{2x}{(x^2+1)(x^2+3)} dx$ if $f(3) = \frac{1}{2}[\ln 5 - \ln 6]$, then $f(4)$ is:

A. $\frac{1}{2}[\ln 17 - \ln 19]$
B. $\frac{1}{2}[\ln 19 - \ln 17]$
C. $\ln 19 - \ln 17$
D. $\ln 17 - \ln 19$

Answer (A)

Solution:

$f(x) = \int \frac{2x}{(x^2+1)(x^2+3)} dx$
Let $x^2 = t$
$2xdx = dt$
$\Rightarrow \int \frac{dt}{(t+1)(t+3)}$
$= \frac{1}{2} \int \frac{(t+3)-(t+1)}{(t+1)(t+3)} dt$
$= \frac{1}{2}[\ln|t+1| - \ln|t+3|] + \frac{c}{2}$
$= \frac{1}{2}[\ln|x^2+1| - \ln|x^2+3|] + \frac{c}{2}$
Now $f(3) = \frac{1}{2}[\ln 5 - \ln 6]$
$\Rightarrow \frac{1}{2}[\ln 5 - \ln 6] = \frac{1}{2}[\ln 10 - \ln 12] + \frac{c}{2}$
$\Rightarrow c = 0$
$\therefore f(x) = \frac{1}{2}[\ln|x^2+1| - \ln|x^2+3|]$
$\therefore f(4) = \frac{1}{2}[\ln 17 - \ln 19]$

9. If $f(x) = \int_0^2 e^{|x-t|} dt$, then the minimum value of $f(x)$ is equal to:

A. $2(e-1)$
B. $2(e+1)$
C. $2e-1$
D. $2e+1$

Answer (A)

Solution:

For $x > 2$
$f(x) = \int_0^2 e^{x-t} dt \Rightarrow e^x(-e^{-t})|_0^2 \Rightarrow e^x(1-e^{-2})$
For $x < 0$
$f(x) = \int_0^2 e^{t-x} dt \Rightarrow e^{-x} e^t|_0^2 \Rightarrow e^{-x}(e^2 - 1)$
For $0 \le x \le 2$
$f(x) = \int_0^x e^{x-t} dt + \int_x^2 e^{t-x} dt$

$= -e^x e^{-t}|_0^x + e^{-x}e^t|_x^2$
$= -e^x(e^{-x}-1) + e^{-x}(e^2 - e^x)$
$= -1 + e^x + e^{2-x} - 1$
$= e^{2-x} + e^x - 2$

$$f(x) = \begin{cases} e^x(1-e^{-2})\ x > 2 \\ e^{2-x} + e^x - 2\ 0 \le x \le 2 \\ e^{-x}(e^x - 1)\ x < 0 \end{cases}$$

For $x > 2$
$f(x)_{\min} = e^2 - 1$
For $0 \le x \le 2$
$f'(x) = -e^{2-x} + e^x = 0$
$\Rightarrow e^x = e^{2-x}$
$\Rightarrow e^{2x} = e^2$
$\Rightarrow x = 1$
$f(x)_{min} = 2e - 2 = 2(e-1)$

10. If $f(x) = x^b + 3,\ g(x) = ax + c$. If $\left(g\left(f(x)\right)\right)^{-1} = \left(\frac{x-7}{2}\right)^{\frac{1}{3}}$, then $fog(ac) + gof(b)$ is:

A. 189
B. 195
C. 194
D. 89

Answer (A)

Solution:

$g\left(f(x)\right) = a(x^b + 3) + c$
$\left(g\left(f(x)\right)\right)^{-1} = \left(\frac{x-3a-c}{a}\right)^{\frac{1}{b}} = \left(\frac{x-7}{2}\right)^{\frac{1}{3}}$
$\Rightarrow a = 2$
$\Rightarrow b = 3$
$\Rightarrow c = 1$
$g(x) = 2x + 1$
$f(x) = x^3 + 3$
Now $fog(2) + gof(3) = 128 + 61 = 189$

11. The term independent of x in the expansion of $\left(2x + \frac{1}{x^7} - 7x^2\right)^5$ is :

A. 1372
B. 2744
C. -13720
D. 13720

Answer (C)

Solution:

Using multinomial theorem,
$\left(2x + \frac{1}{x^7} - 7x^2\right)^5$
$= \frac{5!}{\alpha!\beta!\gamma!}(2x)^\alpha \left(\frac{1}{x^7}\right)^\beta (-7x^2)^\gamma$, where $\alpha + \beta + \gamma = 5 \cdots (i)$
$= \frac{5!}{\alpha!\beta!\gamma!} 2^\alpha . (-7)^\gamma x^{\alpha - 7\beta + 2\gamma}$
For independent term,
$\alpha - 7\beta + 2\gamma = 0 \cdots (ii)$
From (i) and (ii), $\beta = \frac{\gamma+5}{8}$
Since α, β, γ are integers from $[1,5]$
$\Rightarrow\ \gamma = 3,\ \beta = 1, \alpha = 1$
$\therefore$ independent term $= \frac{5!}{1!1!3!} 2^1 . (-7)^3$
$= -13720$

12. The value of $A = \begin{bmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 2 & \log_y z \\ \log_z x & \log_z y & 3 \end{bmatrix}$ then $|adj(adj\ A^2)|$ is:

A. 6^4
B. 4^8
C. 4^5
D. 2^8

Answer (D)

Solution:

$$A = \begin{bmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 2 & \log_y z \\ \log_z x & \log_z y & 3 \end{bmatrix}$$
$$|A| = \frac{1}{\log x \log y \log z} \begin{bmatrix} \log x & \log y & \log z \\ \log x & 2\log y & \log z \\ \log x & \log y & 3\log z \end{bmatrix}$$
$$|A| = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$
$\Rightarrow |A| = 2$
$|adj(adj\ A^2)| = |A|^8$
$= 2^8$

13. Sum of two positive integers is 66 and μ is the maximum value of their product $S = \left\{x \in \mathbb{Z},\ x(66 - x) \geq \frac{5\mu}{9}\right\},\ x \neq 0$, then probability of A when $A = \{x \in S; x = 3k,\ x \in \mathbb{N}\}$ is:

A. $\frac{1}{4}$
B. $\frac{2}{3}$
C. $\frac{1}{3}$
D. $\frac{1}{2}$

Answer (C)

Solution:

Let the two numbers be α and β
$\alpha + \beta = 66$
$A.M. \geq G.M.$
$\frac{\alpha+\beta}{2} \geq \sqrt{\alpha\beta}$
$\mu = 33 \times 33 = 1089$
$x(66 - x) \geq \frac{5\mu}{9}$
$x(66 - x) \geq 605$
$x^2 - 66x + 605 \leq 0$
$x \in [11,\ 55]$
Favourable set of values of x for event $A = \{12,\ 15,\ 18,\ \cdots,\ 54\}$
$P(A) = \frac{15}{45} = \frac{1}{3}$

14. Let $L_1 = \frac{x-3}{1} = \frac{y-2}{2} = \frac{z-1}{3}$ and $L_2 = \frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and direction ratios of line L_3 are $< 1,\ -1,\ 3 >$. P and Q are points of intersection of L_1 and L_3 and L_2 & L_3, respectively. Then, distance between P and Q is:

A. $\frac{10}{3}\sqrt{6}$
B. $\frac{8}{3}\sqrt{11}$
C. $\frac{4}{3}\sqrt{11}$
D. $\frac{11}{3}\sqrt{6}$

Answer (B)

Solution:

Let $PQ = AB$
Let $A(3,\ 2,\ 1)$
Equation of line AB:
$\frac{x-3}{1} = \frac{y-2}{-1} = \frac{z-1}{3} = k$ (let)
$\Rightarrow x = kx + 3,\ y = -k + 2,\ z = 3k + 1$
Let coordinates of $B(k+3,\ -k+2,\ 3k+1)$
B lies on L_2
$B(\lambda + 1,\ 2\lambda + 2,\ 3\lambda + 3)$
$k + 3 = \lambda + 1 \Rightarrow \lambda - k = 2$
$2 - k = 2\lambda + 2 \Rightarrow 2\lambda + k = 0 \Rightarrow k = -2\lambda$
$\Rightarrow 3\lambda = 2 \Rightarrow \lambda = \frac{2}{3}$
$B\left(\frac{5}{3}, \frac{10}{3}, 5\right)$
$AB = \sqrt{\left(\frac{4}{3}\right)^2 + \left(\frac{4}{3}\right)^2 + 16}$
$= \frac{4}{3}\sqrt{11} = PQ$

15. If $\vec{a} = -\hat{\imath} + 2\hat{\jmath} + \hat{k}$ is rotated by $90°$ about origin passing through y-axis. If new vector is $\vec{b}$ then projection of $\vec{b}$ on $\vec{c} = 5\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$ is equal to:

A. $\frac{6}{5}$
B. $\frac{3}{5}$
C. $\frac{6}{5\sqrt{3}}$
D. $\frac{6\sqrt{3}}{5}$

Answer (A)

Solution:

$\vec{b} = \lambda\vec{a} + \mu\hat{\jmath}$

$b = \lambda\left(-\hat{\imath} + 2\hat{\jmath} + \hat{k}\right) + \mu\hat{\jmath}$

$\vec{b} \cdot \vec{a} = 0$

$(\lambda\vec{a} + \mu\hat{\jmath})\vec{a} = 0$

$6\lambda + 2\mu = 0$

$\Rightarrow \mu = -3\lambda$

$\vec{b} = \lambda(\vec{a} - 3\hat{\jmath}) = \lambda\left(-\hat{\imath} - \hat{\jmath} + \hat{k}\right)$

$\lambda = \pm\sqrt{2}$

Projection of $\vec{b}$ on $\vec{c} = \left|\vec{b} \cdot \hat{c}\right|$

$= \left|\left(-\hat{\imath} - \hat{\jmath} + \hat{k}\right)\frac{(5\hat{\imath} + 4\hat{\jmath} + 3\hat{k})}{5\sqrt{2}}\right| = \frac{6}{5}$

16. Given $\frac{dy}{dx} = \frac{y}{x}\left(1 + xy^2(1 + \ln x)\right)$. If $y(1) = 3$, then the value of $\frac{y^2(3)}{9}$ is:

A. $-\frac{1}{43 + 27\ln 3}$
B. $\frac{1}{43 + 27\ln 3}$
C. $\frac{9}{59 - 162(1 + \ln 3)}$
D. $\frac{1}{27 - 43\ln 3}$

Answer (B)

Solution:

$\frac{dy}{dx} - \frac{y}{x} = y^3(1 + \ln x)$

$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} - \frac{1}{xy^2} = (1 + \ln x)$

Taking $\frac{1}{y^2} = t$

$\Rightarrow -\frac{2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$

$\therefore -\frac{1}{2}\frac{dt}{dx} - \frac{t}{x} = (1 + \ln x)$

$\Rightarrow \frac{dt}{dx} + \frac{2t}{x} = -2(1 + \ln x)$

I.F. $= e^{\int \frac{2}{x} dx} = x^2$

$\therefore tx^2 = \int -2(1 + \ln x)x^2 dx$

$\Rightarrow tx^2 = -2\left[\frac{(1+\ln x)x^3}{3} - \int \frac{x^2}{3} dx\right] + c$

$\frac{x^2}{y^2} = -2\left[\frac{x^3}{3}(1 + \ln x) - \frac{x^3}{9}\right] + c \cdots (i)$

$y(1) = 3 \Rightarrow \frac{1}{9} = -2\left(\frac{1}{3} - \frac{1}{9}\right) + c$

$\therefore c = \frac{5}{9}$

Now putting $x = 3,\ c = \frac{5}{9}$ in (i)

$\frac{9}{y^2} = -2(9(1 + \ln 3) - 3) + \frac{5}{9}$

$= \frac{59}{9} - 18(1 + \ln 3)$

$\Rightarrow \frac{y^2}{9} = \frac{9}{59 - 162(1 + \ln 3)}$

17. If $a,\ b \in [1,\ 25]$, $a, b \in \mathbb{N}$ such that $a + b$ is multiple of 5, then the number of ordered pair $(a,\ b)$ is ___________.

Answer (125)

Solution:

TYPE	NUMBERS
$5k$	$5,\ 10,\ 15,\ 20,\ 25$
$5k + 1$	$1,\ 6,\ 11,\ 16,\ 21$
$5k + 2$	$2,\ 7,\ 12,\ 17,\ 22$
$5k + 3$	$3,\ 8,\ 13,\ 18,\ 23$
$5k + 4$	$4,\ 9,\ 14,\ 19,\ 24$

$(a,\ b)$ can be selected as

$I.$ 1 of $5k + 1$ and 1 of $5k + 4 = 2 \times 25 = 50$

$II.$ 1 of $5k + 2$ and 1 of $5k + 3 = 2 \times 25 = 50$

$III.$ Both of the type $5k = 25$

Total $= 125$

18. If $\log_2(9^{2\alpha-4} + 13) - \log_2\left(3^{2\alpha-4}.\frac{5}{2} + 1\right) = 2$, then maximum integral value of β for which equation, $x^2 - ((\sum \alpha)^2 x) + \sum(\alpha + 1)^2\beta = 0$ has real roots is ______.

Answer (6)

Solution:

$\log_2(9^{2\alpha-4}+13)-\log_2\left(3^{2\alpha-4}.\frac{5}{2}+1\right)=2$

$\therefore \frac{9^{2\alpha-4}+13}{3^{2\alpha-4}.\frac{5}{2}+1}=4$

Let $3^{2\alpha-4}=t$

$\Rightarrow t^2+13=10t+4$

$\Rightarrow t^2-10t+9=0$

$\Rightarrow t=9,1$

$\Rightarrow \ \alpha=3,2$

Now equation will become:

$x^2-25x+25\beta=0$ has real roots

$\therefore D\geq 0$

$\Rightarrow 25^2-4\cdot 25\beta\geq 0$

$\Rightarrow \beta\leq \frac{25}{4}$

Maximum integral value $=6$